# Method 5 '07

1. (a) (i) Wronksiian $= \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}$

(ii) nonzero

(iii) $W = y_1 y_2' - y_1' y_2$

where $y'' + Py' + Qy = 0$

$$W' = y_1'y_2' + y_1 y_2'' - y_1'' y_2 - y_1'y_2'$$

$$= y_1'y_2' + y_1(-Qy_2 - Py_2')$$
$$-y_1'y_2' - y_2(-Py_1' - Qy_1)$$

$$= -Py_1y_2' + Py_1'y_2$$
$$= -PW$$

$$\Rightarrow \quad W = Ce^{-\int P(x)\,dx}$$

(b) $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin x \quad \leftarrow (*)$$

$m^2 + 5m + 6 = 0$

$\Rightarrow m = -3, -2 \quad \Rightarrow \quad y = Ae^{-3x} + Be^{-2x}$

$(m+3)(m+2) = 0.$

now let $y = A(x)e^{-3x} + B(x)e^{-2x}$

$$y = A(x)e^{-3x} + B(x)e^{-2x}$$

$$y' = A'e^{-3x} - 3Ae^{-3x} + B'e^{-2x} - 2Be^{-2x}$$

Let $A'e^{-3x} + B'e^{-2x} = 0$.

$$\Rightarrow \quad y' = -3Ae^{-3x} - 2Be^{-2x}$$

$$\Rightarrow \quad y'' = -3A'e^{-3x} + 9Ae^{-3x} - 2B'e^{-2x} + 4Be^{-2x}$$

into $(*)$,

$$-3A'e^{-3x} + 9Ae^{-3x} - 2B'e^{-2x} + 4Be^{-2x} - 15Ae^{-3x} - 10Be^{-2x} + 6Ae^{-3x} + 6Be^{-2x} = e^{-2x}\sin x$$

$$-3A'e^{-3x} - 2B'e^{-2x} = e^{-2x}\sin x$$

$$\Rightarrow \quad \begin{aligned} 3A'e^{-3x} + 3B'e^{-2x} &= 0 \\ -3A'e^{-3x} - 2B'e^{-2x} &= e^{-2x}\sin x \end{aligned}$$

$+$

$$B'\cancel{e} = \sin x$$

$$\Rightarrow \quad B = -\cos x$$

and $A'e^{-3x} + B'e^{-2x} = 0$

$$\Rightarrow \quad A' + B'e^{x} = 0$$

$$\Rightarrow \quad A' = -\sin x\, e^{x}$$

$$\Rightarrow \quad A = e^{x}\cos x + \int e^{x}\cos x \, dx$$

$$= e^{x}\cos x + e^{x}\sin x - \int e^{x}\sin x \, dx \quad A$$

$$\Rightarrow \quad A = \frac{e^{x}}{2}(\cos x + \sin x).$$

$\Rightarrow \quad -\frac{e^{-2x}}{2}(\sin x + \cos x)$ is the PI we want.

(c) $\quad y_i = \int e^{xt} f(t)\,dt$

$\Rightarrow \int \left(xt^2 + (5x+1)t + (6x+3)\right) e^{xt} f(t)\,dt$

if this is $\quad \int \frac{d}{dt}(e^{xt} g)\,dt$

$\Rightarrow \quad \int (e^{xt} g' + x e^{xt} g)\,dt$

$\Rightarrow \quad g' = (t+3)f$

$\quad\;\; g = (t^2+5t+6)f$

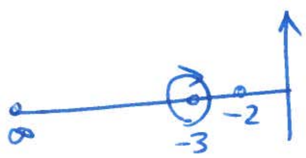
$\Rightarrow \quad \frac{g'}{g} = \frac{t+3}{(t+3)(t+2)} = \frac{1}{t+2}$

$\Rightarrow \quad g = t+2$

$\Rightarrow \quad f = \frac{1}{t+3}.$

Pick $\gamma$ s.t. $\quad [e^{xt} g]_\gamma = 0 \quad$ and $\quad y \neq 0.$

$\Rightarrow \quad t = -2, -\infty \qquad\qquad t = -3.$



$\int_{-\infty}^{-2}$ and $\oint_\gamma = \underline{2\pi i \, e^{-3x}}$

2. $$\frac{dy}{dx} = \frac{Cx+Dy}{Ax+By} = \frac{Q}{P}$$

$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

evals $\begin{vmatrix} \lambda - A & -B \\ -C & \lambda - D \end{vmatrix} = 0$

$$(\lambda - A)(\lambda - D) - BC = 0$$

$$\lambda^2 - (D+A)\lambda + (AD - BC) = 0$$

$$\lambda = \frac{D+A \pm \sqrt{(D+A)^2 - 4(AD-BC)}}{2}$$

$\lambda$ ~~both~~ real. Same sign: node

~~Same~~ different: saddle

complex spiral

imag. centre

+ve real part unstable

−ve ——"—— stable

$$\dot{y} = Cx + Dy$$

$$\dot{x} = Ax + By$$

Consider progression of $x, y$ in time. (point fwd in time)

$$\ddot{x} - \lambda x^2 + x = 0$$

$$\dot{y} = \lambda x^2 - x \quad = Q \quad \longleftarrow \text{ ~~can also~~ integrate to get}$$

$$\dot{x} = y \quad = P \qquad \frac{y^2}{2} = \frac{\lambda x^3}{3} - \frac{x^2}{2} + c$$

as requested in $Q^n$

$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\lambda x - 1 & 0 \end{pmatrix}$$

Singularities where $\dot{x} = \dot{y} = 0$

ie $y = 0, \; x = 0$

$y = 0, \; x = 1/\lambda$

$$J|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{evals} \quad \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

complex imag.

$$J|_{(\frac{1}{\lambda},0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{evals} \quad \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1$$

$\lambda = \pm 1$

real diff sign

$\Rightarrow$ saddle.

Sep's

locally at $(\frac{1}{\lambda}, 0)$

$$\frac{dY}{dX} = \frac{X}{Y}$$

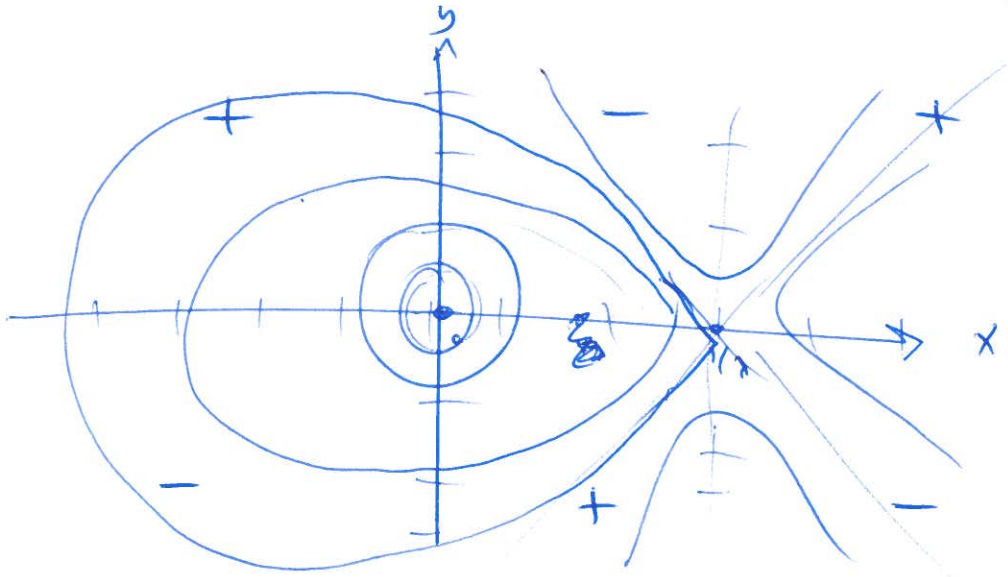
$$Y = mX \Rightarrow \quad m = \frac{1}{m} \Rightarrow m^2 = 1 \quad m = \pm 1.$$

null clines: $\dot{x} = 0 \rightarrow y = 0$

$\dot{y} = 0 \rightarrow \lambda x^2 - x = 0$

$x(x - \frac{1}{\lambda}) = 0$

$x = 0, \frac{1}{\lambda}$




$$\frac{dy}{dx} = \frac{\lambda x^2 - x}{y}$$

at $x = -1, y = 1$ $\quad \frac{dy}{dx} = \frac{\lambda + 1}{1} > 0.$


periodic
Motion under separatrices


$$\frac{y^2}{2} = \frac{\lambda x^3}{3} - \frac{x^2}{2} + c$$

$x = 0$, $y = U$ $\Rightarrow \frac{U^2}{2} = c \Rightarrow \frac{y^2}{2} = \frac{\lambda x^3}{3} - \frac{x^2}{2} + \frac{U^2}{2}$

On separtrix at $(\frac{1}{\lambda}, 0)$

~~Let~~ $\cancel{y = \frac{1}{2}x}$ $\quad \Rightarrow \cancel{\frac{x^2}{2} = \frac{\lambda x^3}{3} + \frac{U^2}{2}}$

$$0 = \frac{1}{3\lambda^2} - \frac{1}{2\lambda^2} + \frac{U^2}{2}$$

$$U^2 = \frac{1}{3\lambda^2}$$

Motion under seps $\Rightarrow U^2 < \frac{1}{3\lambda^2}.$

Period is calculated by

$$T = \int dt = \int \frac{dt}{dy} \, dy = \int \frac{dt}{dy} \frac{dy}{dx} \, dx \, .$$

3.

5. First 2 parts bookwork.

(a) $\displaystyle\int_{-\pi/4}^{\pi/4} e^{-x\sin t} \tan t \, dt$

$f(t) = \tan t \qquad \phi(t) = -\sin t \qquad a = -\pi/4$
$b = \pi/4$

$\phi$ decreasing from a to b.
$\phi'(\pi/4) \neq 0$

$\phi(a) = -\sin(-\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\tan(-\frac{\pi}{4}) = -1$

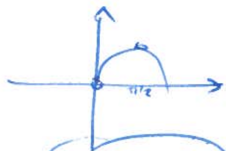
$\phi'(a) = -\cos a$
$= -\frac{\sqrt{2}}{2}$

$\Rightarrow \quad I \sim \; + \dfrac{e^{x\sqrt{2}/2}}{-\sqrt{2}/2 \; x}$

$= -\dfrac{\sqrt{2}}{x} \exp\left(\dfrac{x}{\sqrt{2}}\right).$

(b) $\displaystyle\int_{\pi/6}^{\pi/4} t^3 \sin^x t \, dt$

$= \displaystyle\int_{\pi/6}^{\pi/4} t^3 e^{x \ln\sin t} \, dt$




~~Let~~ $\rightarrow \quad \phi(t) = \ln\sin t$
$f(t) = t^3$

increases
$\phi$ ~~increases~~ from $\pi/6 \rightarrow \pi/4$.

$\phi(b) = \ln\sin(\frac{\pi}{4})$
$= \ln\frac{\sqrt{2}}{2}$

$\phi'(b) = \cot(\frac{\pi}{4})$
$= 1$

$\tan 30° = \frac{\sin 30}{\cos 30} \quad \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad I \sim \cancel{\dfrac{e^{-x\ln 2} \left(\frac{\pi}{6}\right)^3}{\frac{1}{\sqrt{3}} \; x}}$

$= \cancel{e^{-x\ln 2} \cdot \dfrac{\pi^3}{6^3} \dfrac{\sqrt{3}}{x}}$

$I \sim \dfrac{e^{-x\ln(1/\sqrt{2})}}{x}\left(\dfrac{\pi}{4}\right)^3 = \dfrac{\pi^3}{64x} \exp\left(x \ln\dfrac{1}{\sqrt{2}}\right).$

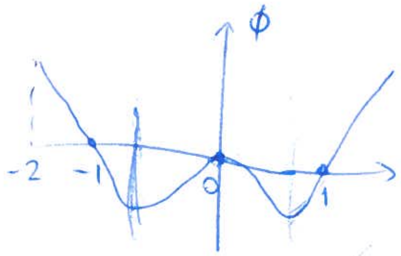
(c) $\int_{-2}^{1} e^{xt^2(t^2-1)}\,dt$

$\varphi''(0)$

$\Rightarrow \phi(t) = t^2(t^2-1)$ $\qquad f(t) = \cancel{0}\,1.$

$t^4-t^2$

$t(4t^2-2)$



$\phi'(t) = 4t^3 - 2t$

$\phi'(t) = 0 \Rightarrow \cancel{2}t(2t^2-1) = 0$

$t = 0 \qquad t = \pm 1/2$

$\int_{-2}^{-1/2} + \int_{-1/2}^{0} + \int_{0}^{1/2} + \int_{1/2}^{1}$

dec. $\qquad$ inc. $\qquad$ dec. $\qquad$ incr.

$\phi(-2) = 4(3) = 12$

$\phi'(-2) = -4\cdot 8 + 2\cdot 2$

$= \cancel{8}\,28$

$\phi(0) = 0$

$\phi'(0) = 0$

$\phi(1) = 0$

$\phi'(1) = 2.$

$\sim \cancel{e}\,\frac{1}{2x}$

$-\frac{e^{12x}}{\cancel{28}\,\cancel{8x}\,24x}$

~~motion~~ dominated by this first bit.

bhr

$\cancel{\int_{-2}^{1} e^{xt^2(t^2-1)}\,dt.}$

~~u = t~~ ~~u = 2~~

$\cancel{u = -2\,t^2(t^2-1)}$

$\cancel{du =}$

$\cancel{u = t^2(t^2-1)}$

$du = (4t^3-2t)\,dt.$

$= 2t(2t^2-1)\,dt$